PROPAGATION OF ELASTIC WAVES IN A ROD IN A LONGITUDINAL MAGNETIC FIELD

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ABSTRACT: The propagation of harmonic waves in discussed for an ideally conducting continuous elastic cylindrical rod within an ideally conducting cylindrical rube. The annulus contains a steady homogeneous longitudinal magnetic field. The dispersion equation is derived. The case of bending vibrations is considered.

1. General. The rod, with radius a, lies in a tube of inside radius b, the field being of strength H. Then $H = \{0, 0, H\}$ in the cylindrical coordinate system r, φ, z. There is no field within the rod. The magnetic field then exerts a pressure on the rod

$$p = {}^1/_8 H^2 / \pi \,. \tag{1.1}$$

The equations for equilibrium give

$$\sigma_{rr}{}^\circ = \sigma_{\varphi\varphi}{}^\circ = -{}^1/_8 H^2 / \pi \,. \tag{1.2}$$

Consider the propagation of harmonic waves such that the displacement is described by the vector

$$\mathbf{u} = \mathbf{U}(r)\, e^{i(\omega t + \nu\varphi + kz)}$$

$$\mathbf{u} = \{u_r, u_\varphi, u_z\}, \qquad \mathbf{U}(r) = \{U(r), V(r), W(r)\}\,. \tag{1.3}$$

The total displacement vector is then $\mathbf{u}' = \mathbf{u}^\circ + \mathbf{u}$. The $^\circ$ denotes the equilibrium value, while the prime denotes the perturbed value, and the quantity without a superscript denotes the small change caused by the perturbation.

The equations of motion are satisfied by the following expressions for the amplitude of the displacement vector [1]:

$$U(r) = A\frac{dJ_\nu(\alpha r)}{dr} + Bk\frac{dJ_\nu(\beta r)}{dr} + C\nu\frac{J_\nu(\beta r)}{r},$$

$$V(r) = Ai\nu\frac{J_\nu(\alpha r)}{r} + Bik\nu\frac{J_\nu(\beta r)}{r} + Ci\frac{dJ_\nu(\beta r)}{dr},$$

$$W(r) = AikJ_\nu(\alpha r) - Bi\beta^2 J_\nu(\beta r),$$

$$\alpha^2 = \frac{\rho\omega^2}{\lambda + 2\mu} - k^2, \qquad \beta^2 = \frac{\rho\omega^2}{\mu} - k^2\,, \tag{1.4}$$

in which A, B, and C are arbitrary constants. Since there is initially no field within the rod, which is ideally conducting, there is no field within the deformed rod.

We put the perturbed magnetic field outside the rod as $H' = H + h$, in which h is a small perturbation produced by the vibration of the rod. As div $H' = 0$ and rot $H' = 0$ in the annulus, we may put $h = -\nabla\Psi$, in which the function Ψ satisfies Laplace's equation $\Delta\Psi = 0$ and so is sought in the form

$$\Psi = \psi(r)\, e^{i(\omega t + \nu\varphi + kz)}\,.$$

The result is

$$\Psi(r) = FiI_\nu(kr) + GiK_\nu(kr), \tag{1.5}$$

in which F and G are arbitrary constants and $I_\nu(kr)$ and $K_\nu(kr)$ are modified Bessel functions.

The following boundary conditions apply at the perturbed surface S' of the rod and at the surface of the tube, as these are ideally conducting: the normal component of H' is zero [2], and the stresses σ_{ij}' within the rod are related to the magnetic pressure $p' = H'^2/8\pi$ at the surface by

$$\mathbf{H}'\cdot\mathbf{n}' = 0, \quad {}^1/_8 H'^2 n_i'/\pi + \sigma_{ij}' n_j' = 0 \text{ on } S'$$

$$H_r' = 0 \quad \text{for} \quad r = b; \qquad i, j = r, \varphi, z, \tag{1.6}$$

in which $\mathbf{n}'$ is the exterior normal to the perturbed surface and n_i' are its components; for the normal $\mathbf{n}'$ we can use the approximate formula

$$\mathbf{n}' = \mathbf{n}^\circ - \nabla u_r,$$

in which u_r is a function only of the coordinates φ and z on S°. On S°

$$\mathbf{H}' = \mathbf{H} - \nabla[FiI_\nu(kr) + GiK_\nu(kr)]\, e^{i(\omega t + \nu\varphi + kz)},$$

$$\frac{H'^2}{8\pi} = \frac{H^2}{8\pi} + \frac{H}{4\pi}k[FI_\nu(kr) + GK_\nu(kr)]\, e^{i(\omega t + \nu\varphi + kz)}$$

$$(r = a)\,. \tag{1.7}$$

We substitute (1.7) into (1.6), express the σ_{ij}' via (1.4), and use (1.2) to get

$$U(r)H + FI_\nu'(kr) + GK_\nu'(kr) = 0,$$

$$\frac{H}{4\pi}k[FI_\nu(kr) + GK_\nu(kr)]\, e^{i(\omega t + \nu\varphi + kz)} + \sigma_{rr} = 0,$$

$$\sigma_{r\varphi} = 0, \qquad {}^1/_8 H^2 iku_r/\pi - \sigma_{zz} = 0 \qquad (r = a);$$

$$FI_\nu'(kb) + GK_\nu'(kb) = 0\,. \tag{1.8}$$

The primes in (1.8) denote derivatives with respect to the arguments of the functions taken at r = a. Then (1.4) allows us to put (1.8) as

$$b_{i1}\frac{A}{a^2} + b_{i2}\frac{B}{a^3} + b_{i3}\frac{C}{a^2} + b_{i4}\frac{F}{a\sqrt{8\pi E}} + b_{i5}\frac{G}{a\sqrt{8\pi E}} = 0$$

$$(i = 1, 2, 3, 4, 5, \ E \text{ — Young's modulus} \tag{1.9}$$

The elements of the determinant $|b_{ij}|$ of system (1.9) in the unknowns A/a^2, B/a^3, C/a^2, $F/a(8\pi E)^{1/2}$, $G/a(8\pi E)^{1/2}$ take the form

$$b_{11} = \frac{H}{\sqrt{8\pi E}}\alpha a J_\nu'(\alpha a), \qquad b_{12} = \frac{H}{\sqrt{8\pi E}}ka\beta a J_\nu'(\beta a),$$

$$b_{13} = \frac{H}{\sqrt{8\pi E}}\nu J_\nu(\beta a), \qquad b_{14} = I_\nu'(ka), \qquad b_{15} = K_\nu'(ka),$$

$$b_{21} = \alpha a J_\nu'(\alpha a) + \left[a^2k^2\left(\frac{\rho\omega^2}{Ek}(1+\varepsilon) - 1\right) - \nu^2\right]J_\nu(\alpha a),$$

$$b_{22} = ka\beta a J_\nu'(\beta a) + ka(\beta^2a^2 - \nu^2)J_\nu(\beta a),$$

$$b_{23} = \nu(J_\nu(\beta a) - \beta a J_\nu'(\beta a)),$$

$$b_{24} = -\frac{2H(1+\varepsilon)}{\sqrt{8\pi E}}akI_\nu(ka),$$

$$b_{25} = -\frac{2H(1+\varepsilon)}{\sqrt{8\pi E}}akK_\nu(ka),$$

$$b_{31} = \nu[\alpha a J_\nu'(\alpha a) - J_\nu(\alpha a)],$$

$$b_{32} = \nu ka[\beta a J_\nu'(\beta a) - J_\nu(\beta a)],$$

$$b_{33} = (\nu^2 - {}^1/_2\beta^2a^2)J_\nu(\beta a) - \beta a J_\nu'(\beta a),$$

$$b_{34} = b_{35} = b_{44} = b_{45} = 0,$$

$$b_{41} = \left(\frac{H^2}{8\pi E} - \frac{1}{1+\varepsilon}\right)ka\alpha a J_\nu'(\alpha a),$$

$$b_{43} = \left(\frac{H^2}{8\pi E} - \frac{1}{2(1+\varepsilon)}\right)\nu ka J_\nu(\beta a),$$

$$b_{42} = \left(\frac{H^2}{8\pi E}k^2a^2 - \frac{k^2a^2}{2(1+\varepsilon)} + \frac{\beta^2a^2}{2(1+\varepsilon)}\right)\beta a J_\nu'(\beta a),$$

$$b_{51} = b_{52} = b_{53} = 0, \qquad b_{54} = I_\nu'(kb), \qquad b_{55} = K_\nu'(kb), \tag{1.10}$$

in which ε is Poisson's ratio.

2. Dispersion equation. The determinant of (1.9) with the elements of (1.10) is equated to zero to give the dispersion equation

$$|b_{ij}| = 0 \qquad (2.1)$$

We put

$$\alpha^2 a^2 = x^2 (y^2 (1 + \varepsilon)(1 - 2\varepsilon)(1 - \varepsilon)^{-1} - 1) = X^2,$$

$$h^2 = {}^1/_8 H^2 / \pi E, \qquad \beta^2 a^2 = x^2 (2y^2 (1 + \varepsilon) - 1) = Y^2,$$

$$y^2 = \rho\omega^2 / k^2 E, \quad x = ka\,;$$

then the elements of (2.1) take the form

$$b_{11} = hXJ_\nu'(X), \qquad b_{12} = hYJ_\nu'(Y), \qquad b_{13} = h\nu J_\nu(Y),$$

$$b_{14} = I_\nu'(x), \qquad b_{15} = K_\nu'(x),$$

$$b_{21} = XJ_\nu'(X) + [x^2(y^2(1+\varepsilon) - 1) - \nu^2] J_\nu(X),$$

$$b_{22} = YJ_\nu'(Y) + (Y^2 - \nu^2) J_\nu(Y),$$

$$b_{23} = \nu [J_\nu(\beta a) - YJ_\nu'(\beta a)], \qquad b_{24} = -2h(1+\varepsilon) xI_\nu(x),$$

$$b_{25} = -2h(1+\varepsilon) xK_\nu(x), \qquad b_{31} = \nu [XJ_\nu'(X) - J_\nu(X)],$$

$$b_{33} = (\nu^2 - {}^1/_2 Y^2) J_\nu(Y) - YJ_\nu'(Y),$$

$$b_{32} = \nu[YJ'_\nu(Y) - J_\nu(Y)], \quad b_{34} = b_{35} = 0,$$

$$b_{41} = (h^2 - 1/1 + \varepsilon) XJ_\nu'(X),$$

$$b_{42} = \left(h^2 + y^2 - \frac{1}{1+\varepsilon}\right) YJ_\nu'(Y),$$

$$b_{43} = \left(h^2 - \frac{1}{2(1+\varepsilon)}\right) \nu J_\nu(Y),$$

$$b_{44} = b_{45} = 0, \quad b_{51} = b_{52} = b_{53} = 0,$$

$$b_{54} = I_\nu'(kb), \qquad b_{55} = K_\nu'(kb)\,. \qquad (2.2)$$

We expand the determinant of (2.1) with the elements of (2.2) with respect to the elements in the last two columns to convert (2.1) to

$$|a_{ij}|\,|c_{ij}|^{-1} + 2h^2(1+\varepsilon)x^2\beta_\nu = 0 \qquad (i, j = 1, 2, 3), \qquad (2.3)$$

$$a_{11} = XJ_\nu'(X) + [x^2(y^2(1+\varepsilon) - 1) - \nu^2] J_\nu(X),$$

$$a_{12} = YJ_\nu'(Y) + (Y^2 - \nu^2) J_\nu(Y),$$

$$a_{13} = \nu [J_\nu(Y) - YJ_\nu'(Y)],$$

$$c_{21} = a_{21} = \nu [XJ_\nu'(X) - J_\nu(X)],$$

$$c_{31} = a_{31} = (h^2 - (1+\varepsilon)^{-1}) XJ_\nu'(X),$$

$$c_{22} = a_{22} = \nu [YJ_\nu'(Y) - J_\nu(Y)],$$

$$c_{23} = a_{23} = (\nu^2 - {}^1/_2 Y^2) J_\nu(Y) - YJ_\nu'(Y),$$

$$c_{32} = a_{32} = \left(h^2 + y^2 - \frac{1}{1+\varepsilon}\right) YJ_\nu'(Y),$$

$$c_{33} = a_{33} = \left(h^2 - \frac{1}{2(1+\varepsilon)}\right) \nu J_\nu(Y),$$

$$c_{11} = hXJ_\nu'(X), \qquad c_{12} = hYJ_\nu'(Y), \qquad c_{13} = h\nu J_\nu(Y),$$

$$\beta_\nu = \frac{K_\nu(ka) I_\nu'(kb) - K_\nu'(kb) I_\nu(ka)}{ka\,[K_\nu'(ka) I_\nu'(kb) - K_\nu'(kb) I_\nu'(ka)]}\,. \qquad (2.4)$$

3. Bending vibrations. We put $\nu = 1$ in (2.3) and (2.4) to get the dispersion equation, reducing (2.3) by the common factor $XJ_1'(X)\,YJ_1'(Y)J_1(Y)$.

$$|b_{ij}|\,|d_{ij}|^{-1} + 2h^2(1+\varepsilon)x^2\beta = 0 \qquad (i, j = 1, 2, 3), \qquad (3.1)$$

$$b_{11} = 1 + [x^2(y^2(1+\varepsilon) - 1) - 1]\,\varphi_1(X),$$

$$b_{12} = 1 + (Y^2 - 1)\,\varphi_1(Y), \qquad b_{13} = 1 - 1/\varphi_1(Y),$$

$$b_{21} = d_{21} = 1 - \varphi_1(X), \qquad b_{22} = d_{22} = 1 - \varphi_1(Y),$$

$$b_{23} = d_{23} = 1 - {}^1/_2 Y^2 - 1/\varphi_1(Y),$$

$$b_{31} = d_{31} = h^2 - 1/(1+\varepsilon),$$

$$b_{32} = d_{32} = h^2 + y^2 - 1/(1+\varepsilon),$$

$$b_{33} = d_{33} = h^2 - {}^1/_2(1+\varepsilon),$$

$$d_{11} = d_{12} = d_{13} = 1, \qquad \varphi_1(\xi) = J_1(\xi) / \xi J_1'(\xi),$$

$$\beta = \frac{K_1(ka)}{kaK_1'(ka)} \frac{[1 - K_1'(kb) I_1(ka) / K_1(ka) I_1'(kb)]}{[1 - K_1'(kb) I_1'(ka) / K_1'(ka) I_1'(kb)]}\,. \qquad (3.2)$$

Since $b > a$, $\beta < 0$ for all values of ka and kb.

For long waves ($x \ll 1$), $\varphi_1(\xi) \approx 1 + \xi^{2/4}$. Since

$$\frac{K_1(\xi)}{\xi K_1'(\xi)} = -\left[1 + \frac{K_0(\xi)}{K_1'(\xi)}\right],$$

we replace the Bessel functions for small values of the argument by

$$K_0(\xi) \approx -(\ln {}^1/_2\xi + C), \qquad K_1(\xi) \approx 1/\xi, \qquad I_1(\xi) \approx {}^1/_2\xi,$$

which gives for β

$$\beta = -\left[1 + k^2a^2\left(\ln\frac{ka}{2} + C\right)\right]\frac{1 + a^2/b^2}{1 - a^2/b^2}, \qquad (3.3)$$

in which $C \approx 0.577$ is Euler's constant. For long wavelengths, (3.1) becomes

$$\frac{\rho\omega^2}{Ek^2} = \frac{1}{4}a^2k^2 + \frac{H^2}{8\pi E}\left\{1 + 2\left[1 + k^2a^2\left(\ln\frac{ka}{2} + C\right)\right]\frac{1 + a^2/b^2}{1 - a^2/b^2}\right\}. \qquad (3.4)$$

Since $\beta < 0$, the rod is always stable in the presence of bending modes.

REFERENCES

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